Network Computing and Efficient Algorithms

Maximal Independent Set

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Outline

Maximal Independent Set

Definition 7.1 (Independent Set)

Given an undirected Graph $G = (V, E)$ an independent set is a subset of nodes $U \subseteq V$,such that no two nodes in U are adjacent. An independent set is a **maximal** if no node can be added without violating independence. An independent set of **maximum** cardinality is called maximum.

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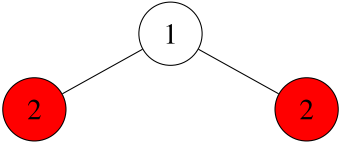

Figure 7.2: Example graph with 1) a maximal independent set (MIS) and 2) a maxinum independent set (MaxIS).

Remarks

- Computing a maximum independent set (MaxIS) is equivalent to computing maximum clique on the complementary graph.
  - Both problems are NP-hard.
  - Not approximable within $n^{0.5-\varepsilon}$ within polynomial time.
- MIS and MaxIS can be quite different.
- Computing a MIS sequentially is trivial...
  - How to compute a MIS in a distributed way?

Slow MIS

Require: Node IDs
Every node *v* executes the following code:

ALGORITHM 7.3: SLOW MIS()

1: **if** all neighbours of *v* with lager identifiers have decided not to join the MIS **then**
2: *v* decides to join the MIS

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Theorem 7.4 (Analysis of Algorithm 7.3).

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Not better than the sequential algorithm in the worst case!

Using Vertex Coloring

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  - In the first round all nodes of the first color join the MIS and notify their neighbors.
  - Then, all nodes of the second color which do not have a neighbor that is already in the MIS join the MIS and inform their neighbors.
  - This process is repeated for all colors.

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Corollary 7.5

Given a coloring algorithm that runs in time T and needs C colors we can construct a MIS in time $T + C$

Fast MIS

The algorithm operates in synchronous rounds, grouped into phases.
A single phase is as follows:

ALGORITHM 7.6: FAST MIS()

1: Each node v marks itself with probability $\frac{1}{2d(v)}$, where $d(v)$ is the current degree of v.
2: If no higher degree neighbour of v is also marked, node v joins the MIS. If a higher degree neighbor if v is marked, node v unmarks itself again (If the neighbors have the same degree, tries a broken arbitrarily, *e.g.*, by identifier).
3: Delete all nodes that joined the MIS and their neighbors, as they cannot join the MIS anymore.

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Theorem 7.11 (Analysis of Algorithm 7.6)

Algorithm 7.6 terminates in expected time $O(\log n)$

Fast MIS terminates in expeded time $O(\log n)$

$d(v)$: node degree of v
$N(v)$: set of neighbors of v
$H(v)$: set of neighbors of v with higher degree …
M: set of nodes marked in Step 1.

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$$\begin{aligned} P[v \notin MIS|v \in M] &= P[there\ is\ a\ node\ w \in H(v), w \in M|v \in M] \\ &= P[there\ is\ a\ node\ w \in H(v), w \in M] \\ &\leq \sum_{w \in H(v)} P[w \in M] = \sum_{w \in H(v)} \frac{1}{2d(w)} \\ &\leq \sum_{w \in H(v)} \frac{1}{2d(v)} \leq \frac{d(v)}{2d(v)} = \frac{1}{2} \end{aligned}$$

Then

$$P[v \in MIS] = P[v \in MIS|v \in M] \cdot P[v \in M] \geq \frac{1}{2} \cdot \frac{1}{2d(v)}$$

Fast MIS terminates in expeded time $O(\log n)$

Lemma 7.8(Good Nodes).

A node v is called good if

$$\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \frac{1}{6}$$

where $N(v)$ is the set of neighbors of v. Otherwise we call v a bad node. **A good node will be removed in Step 3 with probability $p \geq \frac{1}{36}$.**

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Lemma 7.9(Good Edges).

An edge $e = (u, v)$ is called bad if both u and v are bad; else the edge is called good. The following holds:**At any time at least half of the edges are good**

Fast MIS v2

The algorithm operates in synchronous rounds, grouped into phases.
A single phase is as follows:

ALGORITHM 7.12: FAST MIS V2()

1: Each node v chooses a random value $r(v) \in [0, 1]$ and send it to its neighbors.
2: if $r(v) < r(w)$ for all neighbors $w \in N(v)$, node v enters the MIS and informs its neighbors.
3: if v or a neighbor of v entered the MIS, v terminates (v and all edges adjacent to v are removed from the graph), otherwise v enters the nest phase.

Corollary 7.18(Running Time of Algorithm 7.12)

Algorithm 7.12 terminates $w.h.p.$ in $O \log n$ time.

Application for Matching

Definition 7.19 (Matching)

Given a graph $G = (V, E)$ a matching is a subset of edges $M \subseteq E$, such that no two edges in M are adjacent(*i.e.*, where no node is adjacent to two edges in the matching). A matching is **maximal** if no edge can be added without violating the above constraint. A matching is **maximum** cardinality is called maximum. A matching is called **perfect** if each node is adjacent to an edge in the matching

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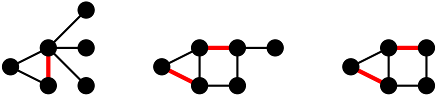

maximal

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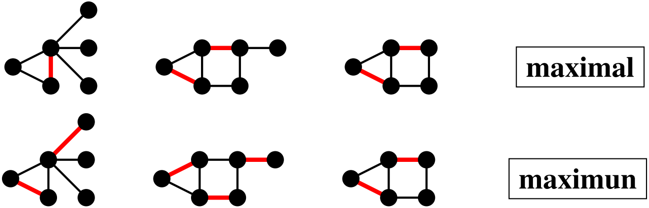

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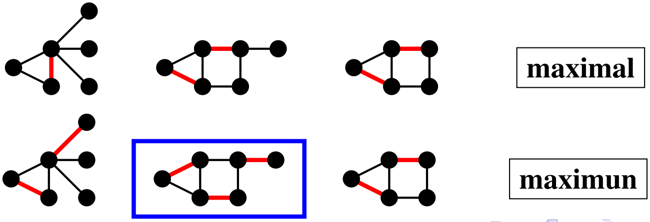

Application for Matching

- In constrast to MaxIS, a maximum matching can be found in polynomial time and also easi to approximate, since any maximal matching is a 2-approximation.
- An independent set algorithm is also a matching algorithm: Let $G = (V, E)$ be the graph for which we want to construct the matching. The so-called line graph G' is defined as follows: for every edge in G there is a node in G'; two nodes in G' are connected by an edge if their respective edges in G are adjacent. A (maximal) independent set in the line graph G' is a (maximal) matching in the original graph G, and vice versa. Using Algorithm 7.12 directly produces a $O(\log n)$ bound for maximal matching.

Application for Coloring

ALGORITHM 7.20 GENERAL GRAPH COLORING()

1: Given a graph $G = (V, E)$ we virtually build a graph $G' = (V', E')$ as follows:
2: Every node $v \in V$ clones itself $d(v) + 1$ times $(v_0, \ldots, v_{d(v)} \in V')$,$d(v)$ being the degree of v in G.
3: The edge set E' of G' is as follows:
4: First all clones are in a clique: $(v_i, v_j) \in E'$, for all $v \in V$ and all $0 \leq i \leq j \leq d(v)$.
5: Second all i^{th} clones of neighbors in the original graph G are connected:$(u_i, v_i) \in E'$, for all $(u, v) \in E$ and all $0 \leq i \leq min(d(u), d(v))$.
6: Now we simply run (simulate) the fast MIS algorithm 7.12 on G'.
7: if node v_i is in the MIS in G', then node v gets color i.

Application for Coloring

Theorem 7.21(Analysis of Algorithm 7.20).

Algorithm 7.20 $(\Delta+1)$-colors an arbitrary graph in $O(\log n)$ time, with high probability, Δ being the largest degree in the graph.

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Proof: Thanks to the clique among the clones at most one clone is in the MIS. And because of the $d(v)+1$ clones of node v every node will get a free color! The running time remains logarithmic since G' has $O(n^2)$n nodes and the exponent becomes a constant factor when applying the logarithm.

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- Together with Corollary 7.5 we get quite close ties between $(\Delta+1)$-coloring and the MIS problem.
- Computing a MIS also solves another graph problem on graphs of bounded independence.

Application for Dominating Sets

Definition 7.22(Bounded Independence).

$G = (V, E)$ is of bounded independence, if for every node $v \in V$ the largest independent set in the neighborhood $N(v)$ is bounded by a constant.

Definition 7.22((Minimum) Dominating Sets).

A dominating set is a subset of the nodes such that each node is the set or adjacent to a node in the set.
A minimum dominating set is a dominating set containing the least possible number of nodes.

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A minimum dominating set is a dominating set containing the least possible number of nodes.

- In general, finding a dominating set less than a factor $\log n$ larger than a minimum dominating set is NP-hard.
- Any MIS is a dominating set: if a node was not covered, it could join the independent set.
- In general a MIS and a minimun dominating set have not much in common (think of a star). For graphs of bounded Independence, this is different.

Application for Dominating Sets

Corollary 7.24.

On graphs of bounded independence, a constant-factor approximation to a minimum dominating set can be found in the time $O(\log n)$ *w.h.p.*

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Proof: Denote by M a minimum dominating set and by I a MIS. Since M is a dominating set, each node from I is in M or adjacent to a node in M. Since the graph is of bounded independence, no node in M is adjacent to more than constantly many nodes from I. Thus, $|I| \in O(|M|)$. Therefore, we can compute a MIS woth Algorithm 7.12 and output it as the dominating set, which takes $O(\log n)$ rounds $w.h.p.$

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